Magnetic Black Holes Are Also Unstable

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Most black holes are known to be unstable to emitting Hawking radiation (in asymptotically flat spacetime). If the black holes are non-extreme, they have positive temperature and emit thermally. If they are extremal rotating black holes, they still spontaneously emit particles like gravitons and photons. If they are extremal electrically charged black holes, they are unstable to emitting electrons or positrons. The only exception would be extreme magnetically charged black holes if there do not exist any magnetic monopoles for them to emit. However, here we show that even in this case, vacuum polarization causes all magnetic black holes to be unstable to emitting smaller magnetic black holes.

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I. INTRODUCTION

Hawking [1, 2] showed that black holes in asymptotically flat spacetime are unstable to emitting thermal radiation at temperature

$$T = \frac{\kappa}{2\pi} = \frac{\sqrt{M^2 - Q^2 - J^2/M^2}}{2\pi(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - J^2/M^2})}, \quad (1)$$

where κ is the horizon surface gravity, M is the mass, Q is the charge, and J is the angular momentum, using Planck units in which $\hbar = c = G = k_{\rm Boltzmann} = 1/(4\pi\epsilon_0) = 1$.

This Hawking radiation decreases the mass and angular momentum of the hole at the following rate:

$$-\frac{d}{dt}\begin{pmatrix} M \\ J \end{pmatrix} = \sum_{j,l,m,p} \int_{\mu_j}^{\infty} \frac{d\omega}{2\pi} \begin{pmatrix} \omega \\ J \end{pmatrix} \times \frac{\Gamma_{jlmp}(\omega)}{\exp\left[(\omega - e\Phi - m\Omega)/T\right] \mp 1}. \quad (2)$$

Here the sum is over the species (labeled by j), the total angular momentum number l of each wave mode, the axial angular momentum number m of the mode, and the polarization p of the mode, and the integral is over the frequency ω of the mode, from the rest mass μ_j of the species to infinity.

In the exponential in the denominator, e represents the charge of the particle being emitted, Φ is the electrostatic potential of the hole, and Ω is the angular velocity of the hole. The upper sign after the exponential ($-$) is for bosons, and the lower sign ($+$) is for fermions. The absorption coefficient is

$$\Gamma_{jlmp}(\omega) = 1 - A_{jlmp}(\omega), \quad (3)$$

where $A_{jlmp}(\omega)$ is the classical amplification coefficient for the mode, the ratio of the outgoing to the ingoing flux at spatial infinity for the mode with the boundary condition of ingoing group velocity at the black hole horizon. (The absorption coefficient is nonnegative except for bosonic superradiant modes that have a negative exponent in the exponential so that the denominator is negative, and then the absorption coefficient is negative so that the Hawking radiation in each mode drains energy from the hole.)

Now we can see that there are various special cases.

(1) A Schwarzschild black hole, with $Q = J = \Phi = \Omega = 0$, emits thermal radiation with temperature $T = 1/(2\pi M)$.

(2) An uncharged extreme rotating Kerr black hole, with $Q = \Phi = 0$ but $J^2 = M^4$, has $T = 0$ but emits at a nonzero rate in the energy range $\mu_j < \omega < m\Omega$ where the exponent of the exponential is $-\infty$:

$$-\frac{d}{dt}\begin{pmatrix} M \\ J \end{pmatrix} = \sum_{j,l,m,p} \int_{\mu_j}^{m\Omega} \frac{d\omega}{2\pi} \begin{pmatrix} \omega \\ m \end{pmatrix} \left[\mp\Gamma_{jlmp}(\omega)\right]. \quad (4)$$

For bosons, this is the spontaneous emission corresponding to the stimulated emission that is a quantum description of superradiant amplification, $A_{jlmp}(\omega) > 1$. For fermions, there is an analogous spontaneous emission even though the Pauli exclusion principle prevents the amplification factor from being greater than unity (as Richard Feynman explained to William Press, Saul Teukolsky, and one of the authors (D.N.P.) around 1972 while drawing diagrams on a blackboard and saying, "I'm supposed to be good at these diagrams"). Therefore, even though there is no true superradiance for fermions, one can say that there is a "superradiant" range for each where $\omega - e\Phi - m\Omega < 0$ and hence where $\exp\left[(\omega - e\Phi - m\Omega)/T\right] = 0$ for $T = 0$.

(3) An extreme charged nonrotating Reissner-Nordstrom black hole, with $Q = M$, $\Phi = 1$, and

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$J = \Omega = 0$, also has $T = 0$ but emits charged particles of the same sign of charge as the hole (say positive for concreteness) at the rate

$$-\frac{d}{dt}\begin{pmatrix} M \\ J \end{pmatrix} = \sum_{j,l,m,p} \int_{\mu_j}^{e\Phi} \frac{d\omega}{2\pi} \begin{pmatrix} \omega \\ m \end{pmatrix} [\mp \Gamma_{jlmp}(\omega)] \,. \quad (5)$$

Henceforth we shall have little use for the axial angular momentum, so let us instead use m for the rest mass μ_j of the species of particles being emitted. For an extreme Reissner-Nordstrom black hole of mass M and charge $Q = M$, particles of mass m and charge e are emitted within the "superradiant" range

$$m < \omega < e\Phi = \frac{eQ}{r_+} = \frac{eQ}{M + \sqrt{M^2 - Q^2}} = e = \sqrt{\alpha}, \quad (6)$$

where the last equality applies for an elementary particle with the charge of the positron, which in Planck units is the square root of the electromagnetic fine structure constant $\alpha \approx 1/137.036$.

Thus extreme Reissner-Nordstrom black holes can emit particles with $e/m > 1$. This is true for all known elementary charged particles, with the positron having $e/m \approx 2.04 \times 10^{21}$, more than 21 orders of magnitude larger than unity.

In summary, it is known that all black holes that are neutral or have just ordinary electric charge are unstable to losing mass by Hawking emission. However, we must examine the case of extremal black holes with magnetic charge.

II. EMISSION OF ORDINARY MAGNETIC MAGNETIC MONOPOLES

If magnetic monopoles with magnetic charge

$$g = \frac{n}{2e} \quad (7)$$

(the Dirac quantization condition with integer n) and mass $m < g$ exist, they can be emitted from an extreme magnetic black hole with magnetic charge $P = M$, using P to denote the magnetic charge of the black hole in Planck units.

GUTs generally predict magnetic monopoles with masses

$$m \sim \frac{M_G}{e^2} < g \sim \frac{1}{e}, \quad (8)$$

since the GUT unification scale M_G generally is significantly more than an order of magnitude lower than the Planck mass $M_{\rm Pl} \equiv 1$, $M_G < e = \sqrt{\alpha} \sim 0.1 = 0.1 M_{\rm Pl}$. If these magnetic monopoles exist, extreme magnetic black holes would be unstable to emitting these magnetic monopoles with magnetic charge g greater than their mass m.

But what if there are no GUT monopoles? If there are no GUT monopoles, one might think that extreme magnetic black holes would be stable. Classically, they would have magnetic charge

$$P = \frac{N}{2e} \quad (9)$$

and mass $M = P$. No kinetic energy would be released if a large $P = M$ hole split into smaller holes with $\sum M_i = \sum P_i = P$. Thus, there would be no phase space available for this putative decay.

If the large extreme hole did split into smaller extreme holes, these smaller holes (when not moving relative to each other, which would be the case since there is no kinetic energy released) would have no forces between them, since the attractive gravitational forces would be precisely balanced by the repulsive magnetic forces:

$$-F_{\rm grav} = \frac{M_1 M_2}{r^2} = F_{\rm mag} = \frac{P_1 P_2}{r^2}. \quad (10)$$

III. ORIGIN OF THE QUANTUM INSTABILITY OF A MAGNETIC BLACK HOLE

Vacuum polarization gives $M < P$ and makes extreme magnetic black holes unstable to splitting.

That is, the mass-to-charge ratio of an extremal magnetic charged black hole is reduced below unity by vacuum polarization:

$$\mathcal{E}(P) \equiv \frac{M_{\rm extreme}}{P} = 1 - \delta(P) < 1, \quad (11)$$

where vacuum polarization gives a positive $\delta(P)$ that increases with the magnetic field strength at the horizon,

$$B_+ = \frac{P}{r_+^2} \approx \frac{1}{P}. \quad (12)$$

Hence smaller holes, with smaller P, have bigger B_+, bigger $\delta(P)$, and smaller $\mathcal{E}(P)$. Thus kinetic energy is released when a large extreme black hole splits into smaller ones.

One might object that since the entropy of an extreme black hole is

$$S = \frac{1}{4}A = \pi r_+^2 \approx \pi P^2, \quad (13)$$

entropy would be reduced when a large extremal black hole with magnetic charge P splits up into smaller extremal black holes with $\sum P_i = P$. However, in asymptotically flat spacetime with the positive kinetic energy released by the splitting, there is an infinite volume of phase space available and hence an infinite capacity for entropy in the form of the positions and momenta of the final black holes. Thus there is no restriction from the second law of thermodynamics against a large black hole

splitting into smaller ones in asymptotically flat spacetime, though there would be for a space of finite volume or for a space of effectively finite volume, such as anti-deSitter spacetime (a problem which shall be left for the future).

IV. MASS-TO-CHARGE RATIO OF EXTREMAL BLACK HOLES

The mass-to-charge ratio of extremal black holes, $\mathcal{E}(P)$ given by Eq. (11), is shifted below unity by the effects of the vacuum polarization of charged particle fields around the magnetically charged black hole.

The largest effect will be by the charged particle field with the lowest mass, the electron-positron field. Therefore, consider the one-loop effect of the electron-positron field on the vacuum. This is given by the 1936 Euler-Heisenberg Lagrangian [3] and dominates for weak fields.

Define the quantity

$$b \equiv \frac{eB}{m^2} \tag{14}$$

which is dimensionless even without setting $G = 1$, where m and e are the mass and charge of the positron. Then for a uniform magnetic field B, the Lagrangian density through one loop in the electron-positron field is

$$\begin{aligned} L &= -\frac{B^2}{8\pi}[1 - \frac{e^2}{\pi} I(b)], && (15)\\ I(b) &= \int_0^\infty dx F(x) \exp\left(-\frac{x}{b}\right), && (16)\\ F(x) &= \frac{1}{x^3}(1 + \frac{1}{3}x^2 - x\coth x). && (17) \end{aligned}$$

Now we need to ask the question of when the magnetic field is sufficiently homogeneous that the equations above for the Lagrangian density of a uniform field are a good approximation for the Lagrangian density of a non-uniform field. For this, we can use the criteria given by V. I. Ritus [4], for a length scale λ of the inhomogeneity that for a magnetic black hole is the radius r:

$$\begin{aligned} \lambda &= r \gg \min\left(\frac{m}{eB}, \frac{1}{\sqrt{eB}}\right)\\ &= \min\left(\frac{mr^2}{eP}, \frac{r}{\sqrt{eP}}\right) = \min\left(\frac{2mr^2}{N}, \frac{\sqrt{2}r}{\sqrt{N}}\right), \end{aligned} \tag{18}$$

where Eq. (9) gives $N = 2eP$, an integer. That is, the formulas for the Lagrangian density of a uniform magnetic field give a good approximation for the Lagrangian density of the non-uniform field outside an extreme magnetic black hole for

$$N \gg \min(mr, 1). \tag{19}$$

At the horizon, $r = r_+ \approx P = N/(2e)$, so

$$mr_+ \approx \frac{m}{2e}N \approx 2.45 \times 10^{-22} N \ll N, \tag{20}$$

and so $N \gg \min(mr_+, 1)$ is always satisfied for extreme magnetic black holes that are much larger than the Planck size and hence have $N \gg 1$.

$N \gg \min(mr, 1)$ will not be satisfied for $r > (2e/m)r_+ \sim 4\times 10^{21} r_+$, but at this radius the Lagrangian density will be smaller than at the horizon by a factor of $\sim m^2/(2e)^2 \sim 6\times 10^{-44}$, so that the effects of larger radii, where the formulas above are not a good approximation, will be negligible. That is, since the dominant effect of the vacuum polarization is fairly near the horizon, for large extreme magnetic black holes, we can always neglect the (radial) inhomogeneity of the magnetic field.

Now we must calculate the black hole metric with vacuum polarization, using the Euler-Heisenberg Lagrangian (15). The electromagnetic field tensor has the form

$$\mathbf{F} = P \sin\theta d\theta \wedge d\phi = B\widehat{d\theta} \wedge \widehat{d\phi}, \tag{21}$$

where $\widehat{d\theta} = d\theta/r$ and $\widehat{d\phi} = \sin\theta d\phi/r$ are the orthonormal one-forms, and where the orthonormal magnetic field strength is

$$B = \frac{P}{r^2}. \tag{22}$$

Here r is a Schwarzschildean radial coordinate, defined so that surfaces of constant r are two-spheres with area $4\pi r^2$.

The vacuum polarization of electrically charged fields, such as the electron-positron field, does not produce any density of magnetic charge and so does not affect these formulas for the magnetic field, which arise simply from the Maxwell equation $d\mathbf{F} = 0$ in the absence of magnetic monopoles. We are indeed assuming no magnetic monopoles present in the theory (or else they would themselves make the extreme black hole unstable, assuming that they have $m < g$), other than black holes.

Now the magnetic field outside a large extreme magnetic black hole would indeed produce a vacuum polarization of smaller magnetic black holes so that $d\mathbf{F}$ would not quite be zero, or Br^2 would not quite be constant. However, this would be a very tiny effect. Since the mass of these smaller magnetic black holes is so much larger than the mass of electrons and positrons, the effects of their vacuum polarization would be much smaller than that of the electron-positron field. Thus here we shall take $B = P/r^2$ as essentially exact.

With the Euler-Heisenberg Lagrangian density L given above, the stress-energy tensor of the magnetic field and vacuum polarization is

$$T_{\alpha\beta} = Lg_{\alpha\beta} - 2\frac{dL}{d(B^2)} g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}. \tag{23}$$

For a static spherically symmetric metric, this gives the orthonormal components of the stress-energy tensor as

$$\begin{aligned} \rho &\equiv T_{\hat{0}\hat{0}} = -L = \frac{B^2}{8\pi}\left[1 - \frac{e^2}{\pi} I(b)\right], && (24)\\ P_r &\equiv T_{\hat{r}\hat{r}} = +L = -\frac{B^2}{8\pi}\left[1 - \frac{e^2}{\pi} I(b)\right], && (25) \end{aligned}$$

$$\begin{aligned} P_\perp &\equiv T_{\hat\theta\hat\theta} = L - 2B^2\frac{dL}{d(B^2)} \\ &= \frac{B^2}{8\pi}\left[1 - \frac{e^2}{\pi}\left(I + b\frac{dI}{db}\right)\right]. \end{aligned} \tag{26}$$

If $f' \equiv df/dr$, the static spherically symmetric metric

$$\begin{aligned} ds^2 = &- e^{2\psi(r)}\left(1 - \frac{2m(r)}{r}\right)dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1}dr^2 \\ &+ r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \end{aligned} \tag{27}$$

has the Einstein field equations

$$\psi' = \frac{4\pi r}{1 - 2m/r}(\rho + P_r) = 0 \tag{28}$$

here, so $\psi = 0$ with a suitable choice of the time coordinate t, and

$$\begin{aligned} m' &= 4\pi r^2\rho = \frac{1}{2}r^2B^2\left[1 - \frac{e^2}{\pi}I(b)\right] \\ &= \frac{P^2}{2r^2}\left[1 - \frac{e^2}{\pi}I\left(\frac{eP}{m^2r^2}\right)\right]. \end{aligned} \tag{29}$$

As a check on these equations, we can see that if we set $e^2/\pi = \alpha/\pi = 0$ to ignore the vacuum polarization, we would get the classical magnetic Reissner-Nordstrom metric with

$$m(r) = M - \frac{P^2}{2r}, \tag{30}$$

$$V = -g_{00} = g^{rr} \equiv 1 - \frac{2m(r)}{r} = 1 - \frac{2M}{r} + \frac{P^2}{r^2}. \tag{31}$$

To solve for the metric and mass $M = m(r = \infty)$, it is convenient to define a critical magnetic charge

$$\begin{aligned} P_* &\equiv \frac{N_*}{2e} = \frac{e}{m^2} \approx 4.88\times 10^{43} \\ &\approx 2.74\times 10^{27}\mathrm{Wb} \approx (6.34\text{ gigavolts})t_0, \end{aligned} \tag{32}$$

where $t_0 \approx 13.7\,\mathrm{Ga}$ is the current age of the universe, or

$$N_* = 2eP_* = \frac{2e^2}{m^2} \approx 8.33\times 10^{42}, \tag{33}$$

such that an extreme classical black hole of this magnetic charge would have a marginally strong magnetic field at its horizon,

$$b_+ = \frac{eB_+}{m^2} = \frac{eP}{m^2r_+^2} \approx \frac{e}{m^2P} = \frac{e}{m^2P_*} = 1. \tag{34}$$

Now let

$$q \equiv \frac{P_*}{P} = \frac{N_*}{N}, \quad u \equiv \frac{P}{r}, \tag{35}$$

so

$$b = \frac{eB}{m^2} = qu^2. \tag{36}$$

The quantity q is a constant for each extreme magnetic black hole that is a measure of the strength of the field at the horizon. It take the value unity, $q = 1$, for

$$M = M_* \approx P_* \approx 0.534\times 10^6 M_\odot \approx 788\,000\,\mathrm{km} \approx 2.63\,\mathrm{s}. \tag{37}$$

Note that *smaller* extreme holes have *stronger* magnetic fields at their horizons; $M < M_* \Rightarrow b_+ \approx q > 1$.

The quantity u is an inverse radial variable that goes from $u = 0$ at radial infinity to $u_+ \approx 1$ at the horizon. (One would have the value of u at the horizon, u_+, exactly 1 for the classical Reissner-Nordstrom extreme black hole, but when the vacuum polarization is taken into account, u_+ is shifted slightly away from 1, as we shall see.)

Let

$$g(b) = g(qu^2) = \frac{e^2}{\pi}I(b), \tag{38}$$

$$f(b) = f(qu^2) = 1 - g(b), \tag{39}$$

so

$$\rho = \frac{B^2}{8\pi}(1 - g) = \frac{B^2}{8\pi}f. \tag{40}$$

Now define the classically dimensionless mass function

$$\mu(u) \equiv \frac{m(r = P/u)}{P}. \tag{41}$$

Then the Einstein equation

$$\begin{aligned} \frac{dm}{dr} &= 4\pi r^2\rho = \frac{1}{2}r^2B^2[1 - g(b)] = \frac{1}{2}\frac{P^2}{r^2}(1 - g) \\ &= \frac{1}{2}u^2(1 - g) = \frac{1}{2}u^2 f \end{aligned} \tag{42}$$

becomes

$$\frac{d\mu}{du} = -\frac{1}{2}[1 - g(qu^2)] = -\frac{1}{2}f(qu^2), \tag{43}$$

where $f = 1 - g(qu^2)$.

The boundary condition at radial infinity is that $m = M$ there, so

$$\mathcal{E} \equiv \frac{M}{P} = \frac{m_\infty}{P} = \mu_\infty = \mu(u = 0). \tag{44}$$

At the horizon, $V \equiv 1 - 2m/r = 1 - 2\mu u = 0$, so the value of μ at the horizon, μ_+, is

$$\mu_+ = \frac{1}{2u_+}, \tag{45}$$

always using the subscript $+$ to denote the value of a quantity at the horizon.

For an extreme black hole, $dV/dr = 0$ or $dV/du = 0$ at the horizon, which gives the equation

$$\frac{dV}{du} = -2\mu - 2u\frac{d\mu}{du} = -2\mu_+ + u_+ f(qu_+^2) = -\frac{1}{u_+} + u_+ f_+ = 0, \quad (46)$$

or

$$u_+ = [1 - g(qu_+)]^{-1/2}, \quad (47)$$

a parametric equation determining $U_+ = u_+(q)$.

Then one integrates Eq. (43) from $u = u_+$ to $u = 0$ to determine $\mathcal{E} = \mu(u = 0)$:

$$\begin{aligned}
\mathcal{E} &\equiv \frac{M_{\text{extreme}}}{P} = \mu(u=0) = \mu_+ + \int_{u_+}^{0} \frac{d\mu}{du} du \\
&= \mu_+ + \frac{1}{2}\int_0^{u_+} du[1 - g(qu^2)] \\
&= \frac{1}{2}\left[\frac{1}{u_+} + u_+ - \int_0^{u_+} du\, g(qu^2)\right] \\
&= \frac{1}{2}\left[f_+^{1/2} + f_+^{-1/2} - \int_0^{f_+^{-1/2}} du\, g(qu^2)\right], \quad (48) \\
f_+ &= 1 - g(qu_+^2) = 1 - g(qf_+^{-1}) = f_+(q) \\
&\approx 1 - g(q) = 1 - \frac{\alpha}{\pi} I(q). \quad (49)
\end{aligned}$$

One can see that we have $g(q) = O(\alpha)$ and hence $f_+^{1/2} + f_+^{-1/2} = 1 + O(\alpha^2)$. Therefore, to first order in $\alpha = e^2$ (all that is given accurately by the one-loop Euler-Heisenberg Lagrangian), we have

$$\begin{aligned}
\mathcal{E} &= 1 - \frac{1}{2}\int_0^1 g(qu^2) du + O(\alpha^2) \\
&\approx 1 - \frac{\alpha}{2\pi}\int_0^1 I(qu^2) du \equiv 1 - \frac{\alpha}{2\pi} J(q), \quad (50)
\end{aligned}$$

$$\begin{aligned}
\frac{2\pi}{\alpha}\left(1 - \frac{M}{P}\right) + O(\alpha) &= J(q) \equiv \int_0^1 du I(qu^2) \\
&= \int_0^\infty dx F(x) \frac{1}{2}\sqrt{\frac{x}{q}}\,\Gamma\left(-\frac{1}{2}, \frac{x}{q}\right). \quad (51)
\end{aligned}$$

Therefore, the mass-to-charge ratio $\mathcal{E} = M/P$ for an extremal magnetically charged black hole is reduced from 1 (its classical value for the Reissner-Nordstrom magnetic black hole without any vacuum polarization) by an amount that to lowest order in the fine structure constant α is $\alpha/(2\pi)$ times the quantity $J(q)$ which depends on $q = P_*/P$, the ratio of the critical magnetic charge to that of the actual magnetic charge.

Now we are left with the problem of estimating $J(q)$ for various values of q, which can range from arbitrarily small values, for arbitrarily large extremal black holes, to the huge value of $N_* \sim 10^{43}$ for the smallest extremal magnetic charged black hole, with $N = 1$ and $P = 1/(2e) \approx 5.853$ Planck units.

To use Eq. (51) to calculate $J(q)$, it is useful to note that the function $F(x)$ appearing therein and defined by Eq. (17) may alternatively be written as

$$F(x) = \frac{2x}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2(\pi^2 k^2 + x^2)}. \quad (52)$$

Then we get

$$\begin{aligned}
I(b) &= \sum_{k=1}^{\infty}\frac{1}{\pi^2 k^2}\int_0^\infty dw \exp\left(-\frac{\pi k}{b}\sqrt{e^w - 1}\right) \\
&= \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left[-\mathrm{ci}\left(\frac{\pi k}{b}\right)\cos\left(\frac{\pi k}{b}\right) - \mathrm{si}\left(\frac{\pi k}{b}\right)\sin\left(\frac{\pi k}{b}\right)\right] \\
&= \frac{1}{3}[\ln(2b) - \gamma] + \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n + \frac{1}{2b}}\right) \\
&\quad - \sum_{m=1}^{\infty}\frac{2}{(m+1)(m+2)}\sum_{n=1}^{\infty}\left(n + \frac{1}{2b}\right)^{-m} \\
&= \frac{1}{3}\ln b - \frac{1}{3}\kappa \\
&\quad + \sum_{m=1}^{\infty}\frac{2}{(m+1)(m+2)}\sum_{n=1}^{\infty}\left(\frac{1}{n^m} - \frac{1}{(n + \frac{1}{2b})^m}\right), \quad (53)
\end{aligned}$$

where

$$\kappa = \gamma + \ln\pi - \frac{\zeta'(2)}{\zeta(2)} \approx 2.291\,906\,543\,845\,465\,841\,149\,803\,801.$$

From these formulas, we can get asymptotic expressions for $I(b)$ for b very small or very large. For $b \ll 1$, we get the divergent asymptotic series

$$\begin{aligned}
I(b) &\sim -\sum_{n=1}^{\infty}\frac{B_{2n+2}(2b)^{2n}}{n(n+1)(2n+1)} = \frac{b^2}{45}\left(1 - \frac{4}{7}b^2 + \frac{8}{7}b^4\right. \\
&\quad \left. - \frac{160}{33}b^6 + \frac{176\,896}{5005}b^8 - \frac{5120}{13}b^{10} + O(b^{12})\right) \quad (54)
\end{aligned}$$

For $b \gg 1$, we get

$$\begin{aligned}
I(b) &\sim \left(\frac{1}{3} + \frac{1}{b} + \frac{1}{2b^2}\right)\ln b - \frac{1}{3}\kappa + \frac{2 - \ln\pi}{b} \\
&\quad + \frac{3 - 2\gamma + 2\ln 2}{4b^2} + \frac{\pi^2}{72b^3} + O\left(\frac{1}{b^4}\right). \quad (55)
\end{aligned}$$

We can also write various expressions for $J(q)$, going

beyond Eq. (51), such as

$$
\begin{aligned}
J(q) &= \int_0^1 du \int_0^\infty dx \left(\frac{1}{x^3}+\frac{1}{3x}-\frac{\coth x}{x^2}\right)\exp\left(-\frac{x}{qu^2}\right) \\
&= \sum_{k=1}^{\infty}\frac{1}{\pi^2k^2}\int_0^\infty \frac{z\,dz\,\exp(-z)}{z^2+\pi^2k^2/q^2}\sum_{n=1}^{\infty}\frac{L_n^{-1/2}(z)}{n+1} \\
&= \sum_{k=1}^{\infty}\frac{4}{\pi^2k^2}\int_0^\infty \frac{x^3dx}{x^4+\pi^2k^2/q^2}\left[\exp(-x^2)-\sqrt{\pi}x\,\mathrm{erfc}\,x\right] \\
&= \left(\frac{1}{3}-\frac{1}{q}-\frac{1}{6q^2}\right)\ln q-\frac{1}{3}(\kappa+2)+\sqrt{2}\,\zeta\left(\frac{3}{2}\right)\frac{1}{\sqrt{q}} \\
&\quad -\frac{2+4\ln 2+3\ln\pi}{q}+\frac{6\gamma-13-6\ln 2}{36q^2} \\
&\quad +4\sum_{k=2}^{\infty}\frac{\zeta(k)}{k(k+1)(2k+1)}\left(-\frac{1}{2q}\right)^{k+1}. \qquad (56)
\end{aligned}
$$

The last expression for $J(q)$, with the one infinite series and no integrals, converges for $q > 1/2$.

Now we can also get asymptotic formulas for $J(q)$ for both q very small and for q very large. For $q \ll 1$, we get the divergent asymptotic series

$$
\begin{aligned}
J(q) \sim \frac{q^2}{225}\Bigg(&1-\frac{20}{63}q^2+\frac{40}{91}q^4-\frac{800}{561}q^6 \\
&+\frac{176\,896}{21\,021}q^8-\frac{1024}{13}q^{10}+O(q^{12})\Bigg). \qquad (57)
\end{aligned}
$$

For $q \gg 1$, inserting numerical values for the numerical coefficients of the terms before the series in the last expression of Eq. (56) gives

$$
\begin{aligned}
J(q) \sim\ & \frac{1}{3}\ln q - 1.43063551+\frac{3.69445665}{\sqrt{q}} \\
&-\frac{\ln q+8.20677838}{q}-\frac{\ln q}{6q^2}+O\left(\frac{1}{q^3}\right). \qquad (58)
\end{aligned}
$$

A simple fitting function that matches the two leading terms at both $q \ll 1$ and $q \gg 1$ is

$$
\begin{aligned}
\tilde{J}(q) = \frac{1}{6}[&+0.9583958466\ln(1+0.0001314447009q^2) \\
&+0.0416041534\ln(1+0.6379336776q^2)]. \qquad (59)
\end{aligned}
$$

However, this drops down to $0.30608J(q)$ at $q = 26.355$. One can get a fit to within 5-6% accuracy for all q (and fitting with arbitrarily high accuracy for arbitrarily large or small q) by multiplying $\tilde{J}(q)$ by the exponential of a suitable gaussian in $\ln q$:

$$
J(q) \approx \hat{J}(q) = \tilde{J}(q)\exp\{1.185\exp[-0.225(\ln q-\ln 26)^2]\}. \qquad (60)
$$

Remember that $J(q)$ is given precisely by the double integrals of Eqs. (51) and (56), and that to lowest nontrivial order in the fine structure constant α, the effect of the vacuum polarization of the electron-positron field of charge e and mass m (and ignoring the vacuum polarization effects of other fields, which are smaller, at least for sufficiently large black holes that q is not too much larger than unity) gives, by Eqs. (11), (50), (59), and (60),

$$
\mathcal{E} \equiv \frac{M_{\mathrm{extreme}}}{P} \approx 1-\frac{\alpha}{2\pi}J(q) \approx 1-\frac{\alpha}{2\pi}\hat{J}(q), \qquad (61)
$$

where Eqs. (32), (35), and (37) give

$$
\begin{aligned}
q &\equiv \frac{P_*}{P}=\frac{e}{m^2P}=\frac{4.88\times 10^{43}}{P}=\frac{2.74\times 10^{27}\mathrm{Wb}}{P} \\
&= \frac{0.534\times 10^6 M_\odot}{P}=\frac{7.88\times 10^8\,\mathrm{m}}{P}=\frac{2.63\,\mathrm{s}}{P}, \qquad (62)
\end{aligned}
$$

with P being the magnetic charge (and hence approximately the mass M_{extreme}) of the extreme magnetically charged black hole.

These formulas apply only for P large in Planck units, $P \gg 1$. Since these formulas also assume that the vacuum polarization of the electron-positron field dominates, they actually apply with good accuracy only when the analogous q's for heavier charged fields, such as the muon, are small. Taking these fields to have masses at least two orders of magnitude larger than the electron-positron field, so that the corresponding q's are at least four orders of magnitude smaller than that for the electron-positron field, means that the formulas above should be good so long as the q for the electron-positron field is much smaller than about 10^4, or $P \gg 100M_\odot$.

V. SMALL EXTREME MAGNETIC BLACK HOLES

Let us now go to the other extreme, where $q \gg 10^4$ or $P \ll 100M_\odot$. First consider the case of the smallest extreme magnetic black holes.

The minimum value for the magnetic charge $P = N/(2e)$ is $P = 1/(2e)$, for $N = 1$, giving $q = P_*/P = 2e^2/m^2 = 8.33 \times 10^{42}$. Then

$$
J(q) \approx \frac{1}{3}\ln q - \frac{1}{3}(\gamma+\ln\pi-\frac{\zeta'(2)}{\zeta(2)}+2) \approx 31.5123, \qquad (63)
$$

so

$$
\delta(P) = 1-\frac{M}{P} \approx \frac{\alpha}{2\pi}J(q) \approx 0.0365987 \approx \frac{1}{27.3234}, \qquad (64)
$$

giving

$$
\mathcal{E} = \frac{M}{P} = 1-\delta(P) \approx 0.9634013. \qquad (65)
$$

However, this just includes the vacuum polarization effects of the electron-positron field. The one-loop effect of all charged Dirac fields of charge magnitude e_i and mass $m_i \ll 1$ is

$$
1-\mathcal{E} = \delta \approx \frac{1}{6\pi}\sum_i e_i^2\left[\ln\frac{2ee_i}{m_i^2}-\gamma-\ln\pi+\frac{\zeta'(2)}{\zeta(2)}-2\right].
$$

Including all three charged leptons (electrons, muons, and taus) gives

$$J_{\rm leptons} \approx 85.55, \quad \delta_{\rm leptons} \approx 0.09935, \quad 1-\delta_{\rm leptons} \approx 0.90065.$$

Including three colors of the six quark flavors with masses in GeV taken to be (0.003, 0.006, 0.123, 1.25, 4.2, 171) gives

$$J_{\rm quarks} \approx 133.07, \quad \delta_{\rm quarks} \approx 0.15455, \quad 1-\delta_{\rm quarks} \approx 0.84545.$$

Including all of these charged fields gives

$$J \approx 218.62, \quad \delta \approx 0.2539, \quad \mathcal{E} = 1-\delta \approx 0.7461 \approx \frac{M}{P}.$$

Thus if these numerical values are correct and give the dominant vacuum polarization, the smallest extreme magnetic black hole might have a mass that is about 25% less than what the classical Reissner-Nordstrom metric would indicate without vacuum polarization.

Now let us go to somewhat larger extremal black holes, but such that the q's for all of the lepton and quark fields are large. That is, we shall now consider any $N = 2eP \ll e^2/m_{\rm top}^2 \sim 4\times 10^{31}$. Then

$$\begin{aligned} \mathcal{E} &= \frac{M}{P} = 1-\delta \\ &\approx 1 - \frac{1}{6\pi}\sum_i e_i^2\left[\ln\frac{2ee_i}{m_i^2} - \gamma - \ln\pi + \frac{\zeta'(2)}{\zeta(2)} - 2\right] \\ &\sim 0.75 + 0.0031\ln N, \end{aligned} \tag{66}$$

assuming vacuum polarization purely from quarks and leptons. Thus extreme magnetic black holes with $P \ll e/m_{\rm top}^2 \sim 4\times 10^{32} \sim 10^{25}\,{\rm kg} \sim 1\,{\rm cm}$ might have

$$\begin{aligned} M = M(P) &\sim 0.75P + 0.0031P\ln P \\ &\approx (0.75 + 0.0031\ln N)\frac{N}{2e} \\ &\approx (4.4 + 0.018\ln N)N. \end{aligned} \tag{67}$$

If an extreme black hole of magnetic charge $P = N/(2e)$ splits into two extreme holes of charges $P_1 = N_1/(2e)$, $P_2 = N_2/(2e)$, with $N_1 + N_2 = N$, then the energy released into kinetic energy is

$$\begin{aligned} \Delta E &= M(P) - M(P_1) - M(P_2) \\ &\approx \frac{2e}{3\pi}(N\ln N - N_1\ln N_1 - N_2\ln N_2) \\ &\approx 0.018\left(N_1\ln\frac{N}{N_1} + N_2\ln\frac{N}{N_2}\right). \end{aligned} \tag{68}$$

If $N_1 = N-1 \gg 1$, $N_2 = 1$,

$$\Delta E \approx \frac{2e}{3\pi}(1+\ln N). \tag{69}$$

VI. RENORMALIZATION GROUP ESTIMATE

Is it a coincidence that $1-\mathcal{E} = \delta = O(1)$? By a crude renormalization group calculation using the Minimal Supersymmetric Standard Model and the approximation $-\ln m_{\rm proton} \sim -\ln m_{\rm Higgs} \sim -\ln m_{\rm SUSY} \gg -\ln m_{\rm GUT} \sim -\ln m_{\rm Pl} \equiv 0$, one of us showed [5] that

$$-\ln m_{\rm proton} \sim \frac{\pi}{10e^2}.$$

Then if we have $n_l = 3$ leptons with $-\ln m_l$ comparable to $-\ln m_{\rm proton}$, taking only the leading terms in Eq. (66) gives

$$\begin{aligned} \delta_{\rm lepton} &\approx \frac{e^2}{2\pi}J \sim \frac{e^2}{2\pi}\frac{n_l}{3}\ln\frac{2e^2}{m_l^2} \sim \frac{e^2}{2\pi}\frac{n_l}{3}(-2\ln m_l) \\ &\sim \frac{e^2}{2\pi}\frac{n_l}{3}(-2\ln m_{\rm proton}) \sim \frac{e^2}{2\pi}\frac{n_l}{3}\frac{\pi}{5e^2} = \frac{n_l}{30} = \frac{1}{10}, \end{aligned}$$

or $1-\delta_{\rm lepton} \sim 0.9$, which is accidentally extremely close to the value calculated above, 0.90065. For quarks and leptons all of similar $-\ln m_i$, one gets $\delta \sim 4/15$, $1-\delta \sim 11/15 \approx 0.73$ instead of the value $1-\delta \approx 0.75$ estimated above.

Thus it seems to be no accident that $1-\mathcal{E} = \delta = O(1)$, and indeed the value depends mainly on the number of species of charged particles and is rather insensitive to the actual value of the charge e or the fine structure constant α.

VII. DECAY RATES FOR EXTREME BLACK HOLES

What is the decay time for an extreme magnetically charged black hole of magnetic charge $P = N/(2e) \gg 1/(2e)$ and mass $M \approx P$ to emit a minimal extremal hole of $P = g = 1/(2e)$ and mass $m = \mathcal{E}g$?

Ignoring prefactors, the time is $t \sim e^{2I}$ with tunneling action

$$I = \int_{r_1}^{r_2}\sqrt{-p_r^2}\,dr, \tag{70}$$

where the radial momentum p_r is given by

$$0 = g^{\alpha\beta}\pi_\alpha\pi_\beta + m^2 = -V^{-1}\left(E - \frac{gP}{r}\right)^2 + Vp_r^2 + m^2,$$

$$V \approx \left(1-\frac{P}{r}\right)^2, \quad m \le E \le gP/r_+ \approx g.$$

The minimum action I is for $E = g = m/\mathcal{E} > m$:

$$\begin{aligned} 2I_{\rm min} &= 2\pi gP(1-\sqrt{1-\mathcal{E}^2}) \\ &= \frac{\pi P}{e}(1-\sqrt{1-\mathcal{E}^2}) = \frac{\pi N}{2\alpha}(1-\sqrt{1-\mathcal{E}^2}) \\ &\approx 36.78P(1-\sqrt{1-\mathcal{E}^2}) \approx 215.26N(1-\sqrt{1-\mathcal{E}^2}) \\ &\sim 12.33P \sim 72.18N, \end{aligned} \tag{71}$$

with $\mathcal{E} \sim 0.7461$ to get the numerical results on the last line.

The time for the extremal magnetic black hole to decay is then $t \sim e^{2I_{\min}} \sim e^{72.18N} \approx 10^{31.35N}$. For $P = M_{\odot} = 9.137 \times 10^{37}$ or $N = 2eP = 1.561 \times 10^{37}$,

$$t \sim e^{1.13\times 10^{39}} \approx 10^{4.89\times 10^{38}} \approx 10^{10^{38.69}}. \quad (72)$$

This is much, much greater than a googol, but much, much less than googolplex.

If the extreme magnetic black hole emitted magnetic monopoles of $\mathcal{E} = m/g \ll 1$ instead, one would get $t \sim e^{\pi g P \mathcal{E}^2} = \exp\left(\frac{\pi m^2}{g B_+}\right)$, the inverse Schwinger rate for the field strength at the horizon. For a general value of $\mathcal{E}$, one would get $t \sim e^{2I_{\min}}$ with

$$2I_{\min} = \frac{2}{1+\sqrt{1-\mathcal{E}^2}} \frac{\pi m^2}{g B_+}. \quad (73)$$

VIII. ENERGY AND ENTROPY OF NEAR EXTREME BLACK HOLES

From the results of the previous section, near-extreme magnetic holes take $t \sim e^{2I} = e^{aN}$ to emit a minimal hole, with

$$a = \frac{\pi}{2\alpha}(1 - \sqrt{1-\mathcal{E}^2}) \approx 72.18. \quad (74)$$

If we use the results of [6] under the preferred assumption there that no energy eigenstates of a black hole have high degeneracy, in the time it takes a near-extreme magnetic black hole to lose another unit of its magnetic charge, by photon emission it would get down to having excess energy

$$E \equiv M - M_{\text{extreme}}(P) \sim N^{-\frac{1}{13}} t^{-\frac{2}{13}} \sim e^{-11.11N}.$$

Since this gives $E \ll P^{-3}$, if the hole is thermalized, it has temperature $T \approx E \sim e^{-11.11N}$ and entropy

$$\begin{aligned} S &\approx \frac{1}{4}A + \ln(2\pi^2 P^3 E) \approx \frac{1}{4}A - 11.11N + \frac{38}{13}\ln N \\ &\approx \frac{\pi}{4e^2}N^2 - 11.11N \approx 107.62782253N^2 - 11.11N. \end{aligned}$$

For $P = M_{\odot} = 9.137 \times 10^{37}$, $N = 2eP = 1.561 \times 10^{37}$, one gets

$$T \sim e^{-1.734\times 10^{38}} \approx 10^{-7.529\times 10^{37}} \approx 10^{-10^{37.8767}}, \quad (75)$$

$$S \sim \frac{1}{4}A - 1.734\times 10^{38} \approx 2.623\times 10^{76} - 1.734\times 10^{38}. \quad (76)$$

IX. OPEN QUESTIONS FOR FUTURE WORK

(1) What is the effect of the vacuum polarization from all charged particle fields, and not just that from leptons and quarks, on $M(P)$?

(2) What is the effect of the Nielsen-Olesen-Skalozub phase transition [7, 8] for $B > m_W^2/e$? This arises from the fact that the charged vector boson W develops a negative-energy Landau level in the naïve vacuum and thereafter gives an imaginary contribution to the Euler-Heisenberg Lagrangian for the magnetic field in the naïve vacuum, signifying its instability.

(3) Since the one-loop effects are not very small, what are the effects of higher loops?

X. CONCLUSIONS

Vacuum polarization effects of charged particle fields, in the presence of the magnetic field of an extreme magnetically charged black hole, can reduce the mass of the hole below that given by the classical extreme Reissner-Nordstrom metric with no vacuum polarization. Since the reduction in the mass is greater for smaller extreme black holes, which have stronger magnetic fields at their horizons, this makes it energetically favorable for larger extreme magnetic black holes to split up into smaller ones. Thus magnetic black holes are unstable to splitting, even if there are no non-black-hole magnetic monopoles that they can decay into.

In asymptotically flat spacetime, the entropy of the kinetic energy released in the process can exceed the entropy decrease of the holes themselves and make the process thermodynamically allowed. However, for large extremal black holes, there is a very large action barrier, so the decay process is extremely slow. During the decay process, there is plenty of time for the temperature to drop to exponentially tiny values (assuming no incoming radiation), and the entropy of the hole can drop below $A/4$ by an absolute amount that is very large, even though still extremely tiny in comparison with the much greater value of $A/4$.

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